partial differential equations / Navier-Stokes equation
Author: Renzo Diomedi

N+S eq = Continuity + Momentum = $\vec{\nabla}\cdot\vec{V} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ + $\rho\frac{d\vec{V}}{dt} = \rho\vec{g} - \vec{\nabla}p + \vec{\nabla}\cdot\vec{\tau}$

The Continuity equation is conservative , its output is a scalar value. u, v, w lie on x, y, z.

The Momentum equation is non-conservative, not exactly measurable as a scalar field, decomposable in 3 scalar equations, where *u, v, w* are components of the shift vectors $\vec{u}\ \vec{v}\ \vec{w}$ along the axes *x, y, z,* :

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right)=\rho g_x-\frac{\partial p}{\partial x}+\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}\right)=\rho g_y-\frac{\partial p}{\partial y}+\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)$$

$$\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}\right)=\rho g_z-\frac{\partial p}{\partial z}+\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)$$

where ρ = Density , μ = Viscosity , g = standard gravity (acceleration)

the Independent variables are 4 = x, y, z, t

the Dependent variables are 10 , id est, 3 velocity components $u(x,y,z,t)$, $v(x,y,z,t)$, $w(x,y,z,t)$ + the pressure $p(x,y,z,t)$ + 6 dependent variables given by the Stress Tensor: $\tau_{xx}(x,y,z,t)$, $\tau_{yy}(x,y,z,t)$, $\tau_{zz}(x,y,z,t)$, $\tau_{xy=yx}(x,y,z,t)$, $\tau_{xz=zx}(x,y,z,t)$, $\tau_{yz=zy}(x,y,z,t)$

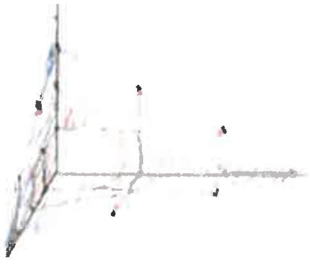
To get the value of the unknowns, i.e. the 10 dependent variables, we have 4 scalar equations only. Momentum equations is also

$$\frac{\partial u}{\partial t}=-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}-w\frac{\partial u}{\partial z}+g_x+\frac{-\frac{\partial p}{\partial x}+\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}}{\rho}$$

$$\frac{\partial v}{\partial t}=-u\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial y}-w\frac{\partial v}{\partial z}+g_y+\frac{-\frac{\partial p}{\partial y}+\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}}{\rho}$$

$$\frac{\partial w}{\partial t}=-u\frac{\partial w}{\partial x}-v\frac{\partial w}{\partial y}-w\frac{\partial w}{\partial z}+g_z+\frac{-\frac{\partial p}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}}{\rho}$$

where the Stress Tensor is



$$\tau_{xx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x} \qquad \tau_{xy}=\tau_{yx}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

$$\tau_{yy}=\frac{\partial v}{\partial y}+\frac{\partial v}{\partial y} \qquad \tau_{xz}=\tau_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}$$

$$\tau_{zz}=\frac{\partial w}{\partial z}+\frac{\partial w}{\partial z} \ \ldots\ldots \qquad \tau_{yz}=\tau_{zy}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}$$

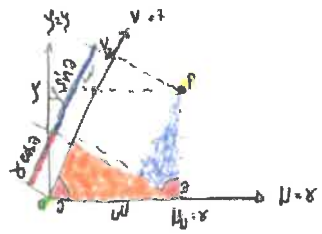
working as

Then the components of the *viscous stress state* are linearly linked to the components of the *deformation velocity* through Stokes' relations:

$$\tau_{xx}=2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\vec{\nabla}\cdot\vec{V}=\frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z}\right) \qquad \tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\tau_{yy}=2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\vec{\nabla}\cdot\vec{V}=\frac{2}{3}\mu\left(\frac{\partial^2 v}{\partial y^2}-\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z}\right) \qquad \tau_{xz}=\tau_{zx}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

$$\tau_{zz}=2\mu\frac{\partial w}{\partial z}-\frac{2}{3}\mu\vec{\nabla}\cdot\vec{V}=\frac{2}{3}\mu\left(\frac{\partial^2 w}{\partial z^2}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\ \ldots\ldots \qquad \tau_{yz}=\tau_{zy}=\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$



A remarkable mathematical tool is the Metric Tensor g . It expresses the property of a geometrically curvable structure having the points of its lattice maintaining the same unchanged distance in relation to the structural components themselves.

μ = metric abscissa , ν = metric ordinate $d\sigma$ = distance of point from origin, whatever the inclination of the reference axes

$d\sigma^2 = d\mu_\mu d\mu^\mu + d\nu_\nu d\nu^\nu = dX_\mu dX^\mu + dX_\nu dX^\nu = dX_i X^j = dX^i X_j = dX_j{}^i$, where $X = reference\ axis\ \mu,\nu$

$i = ith\ \mu,\nu$; $j = jth\ \mu,\nu$, $g_{ij}dX^j = dX_i$, $g^{ij}dX_i = dX^j$, $d\sigma^2 = g^{ij}dX_i dX_j$, $d\sigma^2 = g_{ij}dX^i dX^j$,

contravariant metric tensor: $g^{ij}=\frac{d\sigma^2}{dX_i dX_j}$, covariant metric tensor: $g_{ij}=\frac{d\sigma^2}{dX^i dX^j}$

The 2-dimensional viscous stress metric tensor is :

$$\tau_{\mu\mu}=\mu\left(\frac{\partial^2 u}{\partial^2\sqrt{d\mu^\mu d\mu_\mu}}-\frac{\partial v}{\partial\sqrt{d\nu^\nu d\nu_\nu}}\right)$$

$$\tau_{\nu\nu}=\mu\left(\frac{\partial^2 v}{\partial^2\sqrt{d\nu^\nu d\nu_\nu}}-\frac{\partial u}{\partial\sqrt{d\mu^\mu d\mu_\mu}}\right)$$

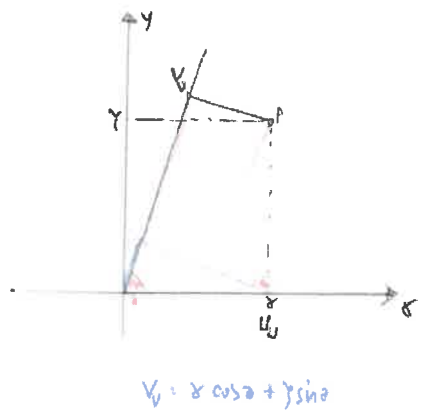
$$\tau_{\mu\nu=\nu\mu}=\mu\left(\frac{\partial u}{\partial\sqrt{d\nu^\nu d\nu_\nu}}+\frac{\partial v}{\partial\sqrt{d\mu^\mu d\mu_\mu}}\right)$$

hypothesize a fluid as incompressible, with variations of its density insignificant and value of its density quasi-constant. The Principal stresses and Shear stresses act in a 3d space. But if we consider them in a space defined by a deformable and curvable sheet, we can suppose a lattice composed of equidistant points subjected to stresses and crossed by a flow to analyze, and that this lattice is deformed by the shift of its axes and its coordinates.

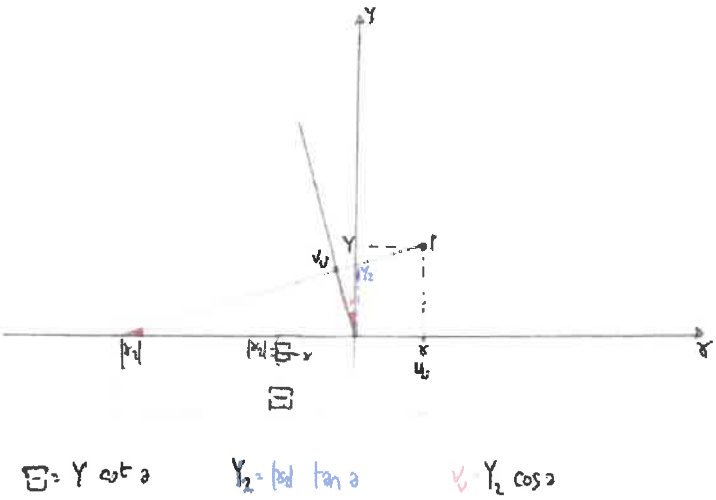
Hypothesize that the Lattice distortions are generated by the Turbulence (Re > 4000), while the points remain at a constant distance beetween them because only the axial coordinates of reference vary. The coordinates of the metric tensor are $\mu_\mu = x$, $y = y$, $v_v = z$ where z is the equivalent of the 3rd axis in a 3d space. So the position of the point in the 3d space is defined by the shift of $v_v = z$ obtained by the variation of the angle beetween x axis and y axis on the 2d-plane. Known the values of x and y and the angle got by

$$2\theta = arctan(tan\,2\theta = s) = \sum_{k=0}^{\infty}(-1)^k \frac{s^{2k+1}}{2k+1} = s - \frac{s^3}{3} + \frac{s^5}{5} - \frac{s^7}{7} + \ \ldots\ldots$$

divided by 2 (the reason of double angle is explaned below), we can get the value of z



Consider a Point that lies in an internal Plane of a Volume crossed by a flow. We must Calculate the variation of the axial angle in the internal plane chosen into the volume, which has value = 90 in the state of quietness, then we can extract a value of the deformation . We need to know the physical properties of the micro volume considered , eg the developed heat, and we must use the co-variant coordinates only.



To get the values of the co-variant coordinates we need to know the interaxle angle.
To get the angle, we need to know the normal and tangential tensions on the axes.
To know these tensions we need to use the Stoke's relations on 2_dimensional viscous stress metric tensor :

$$\sigma_x = \mu\left(\frac{\partial^2 u}{\partial^2 x} - \frac{\partial v}{\partial y}\right)$$

$$\sigma_y = \mu\left(\frac{\partial^2 v}{\partial^2 y} - \frac{\partial u}{\partial x}\right)$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Then we can use the Mohr formulas to calculate the angles of the inclination of the chosen plane into the defined volume:

Sigma = Normal tension
Tau = Tangential tension

$\sigma = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha$ that derived is $\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$

$\tau = -\frac{\sigma_x-\sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$ that derived is $tan\,2\alpha = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$

2α = angle beetween normal axis of the square and the oblique plane + the same angle in the opposite side (note that in Mohr circle , the angle beetween x and y has 180 degrees instead of 90)

then $\tan 2\alpha = -cotan\,2\alpha \rightarrow |\sin 2\alpha| = |\cos 2\alpha| \rightarrow 2\alpha = \pm\frac{\pi}{4}$

then applying this value of this double angle to $\sigma = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha$ and $\tau = -\frac{\sigma_x-\sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$ we get

$$\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2\cos^2 2\alpha + \left(\frac{\sigma_x-\sigma_y}{2}\right)^2\cos^2 - 2\alpha + \tau^2{}_{xy}\sin^2 2\alpha + \tau^2{}_{xy}\sin^2 - 2\alpha + 2\sin 2\alpha\cos 2\alpha\left(\frac{\sigma_x-\sigma_y}{2}\right)\tau_{xy} + 2\sin - 2\alpha\ \cos - 2\alpha\left(\frac{\sigma_x-\sigma_y}{2}\right)\tau_{xy}} =$$

$$= \pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2\frac{1}{2} + \left(\frac{\sigma_x-\sigma_y}{2}\right)^2\frac{1}{2} + \tau^2{}_{xy}\frac{1}{2} + \tau^2{}_{xy}\frac{1}{2} + \sin 4\alpha\left(\frac{\sigma_x-\sigma_y}{2}\right)\tau_{xy} + \sin - 4\alpha\left(\frac{\sigma_x-\sigma_y}{2}\right)\tau_{xy}} =$$

$$= \pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau^2{}_{xy}}$$

Then, the Maximum and the minimum Normal tension is: $\sigma_{\substack{M\\m}} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau^2_{xy}}$

the max tangential tension is: $|\tau| = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$

Then we must choose an appropriate series of interior planes of the volume to calculate the tensions in the different interior points in the volume. Hence we can calculate the amount of energy created over a certain period of time by 2-dimensional version of N-S equations integrated by the 2-d viscous stress metric tensor limited to covariant coordinates only:

Continuity: $\frac{\partial u}{\partial \mu_\mu} + \frac{\partial v}{\partial v_v} = 0$

Momentum :

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial \mu_\mu} - v\frac{\partial u}{\partial v_v} + g_\mu + \frac{-\frac{\partial p}{\partial \mu_\mu} + \mu\left(\frac{\partial^3 u}{\partial^3 \mu_\mu} - \frac{\partial^2 v}{\partial v_v + \partial \mu_\mu}\right) + \mu\left(\frac{\partial^2 u}{\partial^2 v_v} + \frac{\partial^2 v}{\partial v_v + \partial \mu_\mu}\right)}{\rho}$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial \mu_\mu} - v\frac{\partial v}{\partial v_v} + g_v + \frac{-\frac{\partial p}{\partial v_v} + \mu\left(\frac{\partial^2 u}{\partial v_v + \partial \mu_\mu} + \frac{\partial^2 v}{\partial^2 \mu_\mu}\right) + \mu\left(\frac{\partial^3 v}{\partial^3 v_v} - \frac{\partial^2 u}{\partial v_v + \partial \mu_\mu}\right)}{\rho}$$

Now, the dependent variables are 4.

Let a series is composed by the values returned by these equations in their different times. As a Lagrangian system where the particle is the value returned time by time and it makes a series of Eulerian spaces.

This Action is a scalar which is the energy developed in a Time. The total energy is proportional to the minimum area of the worldsheet $d\sigma\, d\vartheta \sqrt{-\det g}$.

This action is local and it must be defined by an Integral as $S = -\frac{T}{c}\int d\sigma\, d\theta \sqrt{-\det g}$ where T = String Tension, c = light speed, but conventionally used as an unit,

where $d\sigma$ = 2D-space defined by covariant coordinates and $d\theta$ is the proper-time, $X^\mu(\sigma,\theta)$, $X^v(\sigma,\theta)$ are functions that determine the worldsheet shape,

X = elements of a space-time vector, in this case $X_1 = t_1$ $X_2 = t_2$, ……..$X_n = t_n$

g is a metric tensor $g = g_{\mu v}\begin{bmatrix} \frac{\partial X^\mu}{\partial \theta}\bullet\frac{\partial X^v}{\partial \theta} & \frac{\partial X^\mu}{\partial \theta}\bullet\frac{\partial X^v}{\partial \sigma} \\ \frac{\partial X^\mu}{\partial \sigma}\bullet\frac{\partial X^v}{\partial \theta} & \frac{\partial X^\mu}{\partial \sigma}\bullet\frac{\partial X^v}{\partial \sigma} \end{bmatrix} = \begin{bmatrix} \dot{X}^2 & \dot{X}\bullet X' \\ X'\bullet\dot{X} & X'^2 \end{bmatrix}$ where $\dot{X} = \frac{\partial X}{\partial \vartheta}$, $X' = \frac{\partial X}{\partial \sigma}$ collects $\partial u, \partial v$, $\det g = \dot{X}^2 X'^2 - (\dot{X}\bullet X')^2$

this metric tensor expands the concept of space by inserting the covariant coordinates.

we do not interested to know the tension of the string composed by elements existing at different times. So the action is $S(t) = -\int_1^n \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}\, d\sigma\, d\theta$, $t \in (1,n)$

we apply it to N-S , i.e. if its elements are the outputs of N-S , the String includes N-S and becomes :

$$S(t) = -\int_1^n \sqrt{\left(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}\right)^2 \cos\alpha^2 - \left(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}\right)^2}\, d\sigma\, d\vartheta \quad , t \in (1,n) \in N$$